Constructions of Lyapunov Functions for Classic SIS, SIR and SIRS Epidemic models with Variable Population Size

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Abstract
In this work we deal with global stability properties of classic SIS, SIR and SIRS epidemic models with constant recruitment rate, mass action incidence and variable population size. The usual approach to determine global stability of equilibria is the direct Lyapunov method which requires the construction of a function with specific properties. In this work we construct different Lyapunov functions for the systems mentioned above using combinations of suitable composite quadratic, simple quadratic and logarithmic functions. And present some examples of the non-uniqueness of Lyapunov functions in epidemic models.

Keywords: Epidemic models, Lyapunov functions, global stability.

1 Introduction
One of the classical problems in mathematical epidemiology is the global stability analysis of equilibria. Traditionally, the Lyapunov function is a powerful tool for the analysis of stability of autonomous systems of differential equations. However it is difficult to construct a Lyapunov function to establish the global stability of equilibria.

This paper presents the construction of Lyapunov functions for SIS, SIR and SIRS epidemic models.

The epidemic models incorporate constant recruitment, disease-induced death and mass action incidence rate.
We prove that the global stability is completely determined by the basic re-
productive number. The global stability of the equilibria is obtained by means
of Lyapunov’s direct method combined with LaSalle’s invariance principle.

Lyapunov functions are not unique, this paper serves to highlight the ac-
tual variation in the overall geometry of different but equally valid Lyapunov
functions.

In this paper we construct Lyapunov functions for each epidemic model using
suitable combinations for the following functions:

The **logarithmic Lyapunov function** proposed by Goh for Lotka-Volterra
systems:

\[ L(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} c_i (x_i - x_i^* - x_i^* \ln \frac{x_i}{x_i^*}), \]

**common quadratic Lyapunov function**, have been extensively exploited for
nonlinear and linear systems:

\[ V(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} c_i^2 (x_i - x_i^*)^2, \]

and we propose the Lyapunov function:

\[ W(x_1, x_2, \ldots, x_n) = \frac{\varepsilon}{2} \left[ \sum_{i=1}^{n} (x_i - x_i^*) \right]^2, \]

we call this, **composite quadratic function**. We applied the composite
quadratic function in the following sections this work.

In [4] prove global stability of the equilibrium of classic SIS, SIR and SIRS
epidemic model using the logarithmic Lyapunov function. In [6] using combina-
tions of common quadratic and logarithmic functions to prove global stability of
quadratic Lyapunov function for epidemic model with delays. We exploited the
suitable combinations of common quadratic, composite quadratic and logari-
thmic functions for SIS, SIR and SIRS epidemic models.

In Sections 2 and 3, we shall construct Lyapunov functions for SIS, SIR
and SIRS epidemic models. In Section 4, we review of Lyapunov functions pro-
posed in the literature of the epidemic model and we construct other Lyapunov
functions to determine global stability of endemic equilibrium.

## 2 The SIS model

Some infections, do not confer any long lasting immunity. Such infections do not
have a recovered state and individuals become susceptible again after infection.
This type of disease can be modelled by SIS type. The total population $N$ is divided into two compartments with $N = S + I$, where $S$ is the number of individuals in the susceptible class, $I$ is the number of individuals who are infectious. The SIS model since one typical pathway is through $S$, then $I$, and then back to $S$, as shown in Figure 1.

\[
\begin{array}{c}
\Lambda \\
\uparrow \\
S \xrightarrow{\beta SI} I \\
\downarrow \\
\mu S \\
\downarrow \\
(\mu + \alpha) I
\end{array}
\]

\textit{Fig. 1. The transfer diagram for the SIS model.}

The transfer diagram leads to the following systems of differential equations for this SIS model are:

\begin{align*}
\frac{dS}{dt} &= \Lambda - \beta SI - \mu S + \phi I, \\
\frac{dI}{dt} &= \beta SI - (\phi + \mu + \alpha)I.
\end{align*}

(1)

The parameters are positive constants. Constant $\Lambda$ is the recruitment rate of susceptibles corresponding to births and immigration, $\mu$ is the per capita natural mortality rate. We assume that a disease may be fatal to some infectious deaths due to disease can be included in a model using the disease-related death rate from infectious class, $\alpha$. And $\phi$ the rate at which infectious individuals and return to susceptible class and they don’t acquire immunity.

Which, together with $N = S + I$, implies

\[N' = \Lambda - \mu N - \alpha I.\]

(2)

Thus the total population size $N$ may vary in time. In the absence of disease, the population size $N$ converges to the equilibrium $\Lambda/\mu$. It follows from (2) that $\lim \sup_{t \to \infty} N \leq \Lambda/\mu$. We thus study (1) in the following feasible region:

\[\Omega = \{(S, I) \in \mathbb{R}_+^2 : S \geq 0, \ I \geq 0, \ S + I \leq \Lambda/\mu\}\]

which can be shown to be positively invariant with respect to (1). Direct calculation shows that system (1) has two possible equilibria in the non-negative triangle $\mathbb{R}_+^2$: the disease-free equilibrium $E^0 = (S^0, I^0) = (\Lambda/\mu, 0)$, and a unique endemic equilibrium $E^* = (S^*, I^*)$ with

\begin{align*}
S^* &= \frac{S^0}{R_0}, \\
I^* &= \frac{\mu(\phi + \alpha + \mu)}{\beta(\alpha + \mu)}(R_0 - 1).
\end{align*}

3
Let $R_0 = \frac{\Lambda \beta}{\mu + \alpha + \mu}$ is often called the basic reproductive number. The $R_0$ has been defined as the average number of secondary infections that occur when one infective is introduced into a completely susceptible host population [2].

2.1 Global stability of disease-free equilibrium

We shall prove that the disease-free equilibrium is global stability by means of Lyapunov function, using combinations of composite quadratic and linear functions.

**Theorem 1.** The disease-free equilibrium $E_0$ of (1) is globally asymptotically stable in $\Omega$ if $R_0 \leq 1$.

**Proof.** Define $U : \{(S, I) \in \Omega : S > 0\} \rightarrow \mathbb{R}$ by

$$U(S, I) = \frac{1}{2} [(S - S^0) + I]^2 + \frac{(\alpha + 2\mu)}{\beta} I.$$  

Then $U$ is $C^1$ on the interior of $\Omega$, $E_0$ is the global minimum of $U$ on $\Omega$, and $U(S^0, I^0) = 0$. The time derivative of $U$ computed along solutions of (1) is

$$U' = \left[ (S - S^0) + I \right] \frac{d(S + I)}{dt} + \frac{(\alpha + 2\mu)}{\beta} \frac{dI}{dt}.$$  

$$= \left[ (S - S^0) + I \right] (\Lambda - \mu(S + I) - \alpha I) + \frac{(\alpha + 2\mu)}{\beta} (\beta SI - (\phi + \mu + \alpha)I).$$  

Using $\Lambda = \mu S^0$ to rewrite this, we get

$$U' = \left[ (S - S^0) + I \right] (\mu S^0 - \mu(S + I) - \alpha I) + \frac{(\alpha + 2\mu)}{\beta} (\beta SI - (\phi + \mu + \alpha)I),$$  

$$= -\mu(S - S^0)^2 - (\mu + \alpha)I^2 - (\alpha + 2\mu) \left( \frac{\phi + \alpha + \mu}{\beta} - S^0 \right) I.$$  

Rewritten $U'$ in term of basic reproductive number, we have

$$U' = -\mu(S - S^0)^2 - (\mu + \alpha)I^2 - \frac{(\alpha + 2\mu)(\phi + \alpha + \mu)}{\beta} (1 - R_0) I.$$  

If $R_0 \leq 1$, then $U' \leq 0$. Note that, $U' = 0$ if and only if $S = S^0$ and $I = 0$, or if $R_0 = 1$, $S = S^0$ and $I = 0$. Therefore the largest compact invariant set in $\{(S, I) \in \Omega : U' = 0\}$ is the singleton $\{E_0\}$, where $E_0$ is the disease-free equilibrium. LaSalle’s invariant principle [5] then implies that $E_0$ is globally asymptotically stable in $\Omega$. 

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2.2 Global stability of endemic equilibrium

In this subsection, we prove the global stability of endemic equilibrium obtained by means of Lyapunov's direct method. The Lyapunov function constructed of the suitable combinations of composite quadratic and logarithmic functions.

**Theorem 2.** If $R_0 > 1$ then the unique endemic equilibrium $E^*$ of (1) is globally asymptotically stable in the interior of $\Omega$.

**Proof:** Define $V : \{(S, I) \in \Omega : S, I > 0\} \to \mathbb{R}$ by

$$V(S, I) = \frac{1}{2} [(S - S^*) + (I - I^*)]^2 + \frac{(\alpha + 2\mu)}{\beta} \left( I - I^* - I^* \ln \frac{I}{I^*} \right).$$

Then $V$ is $C^1$ on the interior of $\Omega$, $E^*$ is the global minimum of $V$ on $\Omega$, and $V(S^*, I^*) = 0$. The time derivative of $V$ computed along solutions of (1) is

$$V' = [(S - S^*) + (I - I^*)] \frac{d(S + I)}{dt} + \frac{\alpha + 2\mu}{\beta} \frac{(I - I^*)}{I} \frac{dI}{dt},$$

$$= [(S - S^*) + (I - I^*)](\Lambda - \mu(S + I) - \alpha I)$$

$$+ \frac{(\alpha + 2\mu)}{\beta} \frac{(I - I^*)}{I}(\beta SI - (\phi + \mu + \alpha)I).$$

Using

$$\Lambda = \mu(S^* + I^*) + \alpha I^*, \quad \beta S^* = (\phi + \mu + \alpha),$$

to rewrite this, we get

$$V' = [(S - S^*) + (I - I^*)]\{-\mu[(S - S^*) + (I - I^*)] - \alpha(I - I^*)\}$$

$$+ (\alpha + 2\mu)(S - S^*)(I - I^*),$$

$$= -\mu(S - S^*)^2 - (\mu + \alpha)(I - I^*)^2.$$  

Hence $V'$ is negative. Note that, $V' = 0$ if and only if $S = S^*$ and $I = I^*$. Therefore the largest compact invariant set in $\{(S, I) \in \Omega : V' = 0\}$ is the singleton $\{E^*\}$, where $E^*$ is the endemic equilibrium. LaSalle's invariant principle [5] then implies that $E^*$ is globally asymptotically stable in the interior of $\Omega$. 

5
3 The SIR and SIRS models

Some infectious disease confer permanent immunity and other diseases confer temporary acquired immunity. These types of diseases can be modelled by SIR and SIRS models, respectively. The total population \( N \) is divided into three compartments with \( N = S + I + R \), where \( S \) is the number of individuals in the susceptible class, \( I \) is the number of individuals who are infectious and \( R \) is the number of individuals recovered.

The SIRS model since one typical pathway is through \( S \), then \( I \), then \( R \), and then back to \( S \), as shown in Figure 2.

\[
\begin{array}{c}
\Lambda \\
S \\
\downarrow \\
\frac{\beta S I}{\mu} \\
\downarrow \\
\frac{\kappa I}{\mu} \\
\downarrow \\
\frac{\gamma R}{\mu} \\
S
\end{array}
\]

Fig. 2. The transfer diagram for the SIRS model.

The transfer diagram leads to the following systems of differential equations for this SIRS model [6] are:

\[
\begin{align*}
\frac{dS}{dt} &= \Lambda - \beta SI - \mu S + \gamma R, \\
\frac{dI}{dt} &= \beta SI - (\kappa + \mu + \alpha)I, \\
\frac{dR}{dt} &= \kappa I - (\mu + \gamma)R,
\end{align*}
\]

(3)

where parameters \( \Lambda, \mu, \beta, \kappa \) and \( \alpha \) are positive constants, and \( \gamma \) a is non-negative constant. Here we assume that \( \kappa \) is the rate at which infectives recover. If the individuals recovered acquired permanent immunity \( \gamma = 0 \) then this SIR model and if \( \gamma \neq 0 \) the individuals acquired temporal immunity, then this SIRS model.

Which, together with \( N = S + I + R \), implies

\[
N' = \Lambda - \mu N - \alpha I.
\]

(4)

Thus the total population size \( N \) may vary in time. In the absence of disease, the population size \( N \) converges to the equilibrium \( \Lambda/\mu \). It follows from (4) that \( \lim_{t \to \infty} N \leq \Lambda/\mu \).

Thus it suffices to consider solutions in region \( \Gamma \):

\[
\Gamma = \{(S, I, R) \in \mathbb{R}_+^3 : S \geq 0, \ I \geq 0, \ R \geq 0, \ S + I + R \leq \Lambda/\mu \}
\]

is positive invariant. As before, the initial value problem is well posed both mathematically and epidemiologically in \( \Gamma \).

The system (3) has a disease-free equilibrium \( E^0 = (S^0, I^0, R^0) = (\Lambda/\mu, 0, 0) \), and a unique endemic equilibrium \( E^* = (S^*, I^*, R^*) \) with coordinates
\[ S^* = \frac{S^0}{R_0}, \]
\[ I^* = \frac{\mu(\mu + \gamma)(\kappa + \alpha + \mu)(R_0 - 1)}{\beta(\kappa \mu + (\mu + \gamma)(\alpha + \mu))}, \]
\[ R^* = \frac{\kappa \mu(\kappa + \alpha + \mu)(R_0 - 1)}{\beta(\kappa \mu + (\mu + \gamma)(\alpha + \mu))}. \]

Here \( R_0 = \frac{\beta \Lambda}{\mu(\kappa + \alpha + \mu)} \) is the basic reproductive number of the system (3).

### 3.1 Global stability of disease-free equilibrium

In this subsection, we prove the global stability of the disease-free equilibrium. The result is obtained by means of Lyapunov function constructed of the combinations of composite quadratic, common quadratic and linear functions.

**Theorem 3.** The disease-free equilibrium \( E^0 \) of (3) is globally asymptotically stable in \( \Gamma \) if \( R_0 \leq 1 \).

**Proof.** Define \( W : \{(S, I, R) \in \Gamma : S > 0\} \to \mathbb{R} \) by

\[
W(S, I, R) = \frac{1}{2}[(S - S^0) + I + R]^2 + \frac{(\alpha + 2\mu)}{\beta} I + \frac{(\alpha + 2\mu)}{2\kappa} R^2.
\]

Then \( W \) is \( C^1 \) on the interior of \( \Gamma \), \( E^0 \) is the global minimum of \( W \) on \( \Gamma \), and \( W(S^0, I^0, R^0) = 0 \). The time derivative of \( W \) computed along solutions of (3) is

\[
W' = \left[(S - S^0) + I + R\right] \frac{d}{dt}(S + I + R) + \frac{(\alpha + 2\mu)}{\beta} \frac{dI}{dt} + \frac{(\alpha + 2\mu)}{\kappa} R \frac{dR}{dt},
\]

\[
= \left[(S - S^0) + I + R\right] (\Lambda - \mu(S + I + R) - \alpha I)
+ \frac{(\alpha + 2\mu)}{\beta} (\beta SI - (\kappa + \mu + \alpha)I)
+ \frac{(\alpha + 2\mu)}{\kappa} R(\kappa I - (\mu + \gamma)R).
\]

Using \( \Lambda = \mu S^0 \) to rewrite this, we get
\[ W' = [(S - S^0) + I + R] (\mu S^0 - \mu (S + I + R) - \alpha I) \\
+ \frac{(\alpha + 2\mu)}{\beta} (\beta SI - (\kappa + \mu + \alpha) I) \\
+ \frac{(\alpha + 2\mu)}{\kappa} (\kappa RI - (\mu + \gamma) R^2), \\
= -\mu[(S - S^0) + R]^2 - (\mu + \alpha) I^2 - \frac{(\alpha + 2\mu)(\mu + \gamma)}{\kappa} R^2 \\
- (\alpha + 2\mu) \left( \frac{(\kappa + \alpha + \mu)}{\beta} - S^0 \right) I. \]

Rewritten \( W' \) in term of basic reproductive number, we have

\[ W' = -\mu[(S - S^0) + R]^2 - (\mu + \alpha) I^2 - \frac{(\alpha + 2\mu)(\mu + \gamma)}{\kappa} R^2 \\
- (\alpha + 2\mu)(\kappa + \alpha + \mu)(1 - R_0) I. \]

If \( R_0 \leq 1 \), then \( W' \leq 0 \). Note that, \( W' = 0 \) if and only if \( S = S^0, I = 0 \) and \( R = 0 \), or if \( R_0 = 1, S = S^0, I = 0 \) and \( R = 0 \). Therefore the largest compact invariant set in \( \{(S, I, R) \in \Gamma : W' = 0 \} \) is the singleton \( \{E^0\} \), where \( E^0 \) is the disease-free equilibrium. LaSalle's invariant principle [5] then implies that \( E^0 \) is globally asymptotically stable in \( \Gamma \).

### 3.2 Global stability of endemic equilibrium

The following result studies the global stability of the endemic equilibrium. The result is obtained by means of Lyapunov function using suitable combinations of composite quadratic, common quadratic and logarithmic functions.

**Theorem 4.** If \( R_0 > 1 \) then the unique endemic equilibrium \( E^* \) of (3) is globally asymptotically stable in the interior of \( \Gamma \).

**Proof:** Define \( L : \{(S, I, R) \in \Gamma : S, I, R > 0 \} \rightarrow \mathbb{R} \) by

\[ L(S, I, R) = \frac{1}{2} [(S - S^*) + (I - I^*) + (R - R^*)]^2 \\
+ \frac{(\alpha + 2\mu)}{\beta} \left( I - I^* - I^* \ln \frac{I}{I^*} \right) + \frac{(\alpha + 2\mu)}{2\kappa} (R - R^*)^2. \]

Then \( L \) is \( C^1 \) on the interior of \( \Gamma \), \( E^* \) is the global minimum of \( L \) on \( \Gamma \), and \( L(S^*, I^*, R^*) = 0 \). The time derivative of \( L \) computed along solutions of (3) is
\[ L' = [(S - S^*) + (I - I^*) + (R - R^*)] \frac{d}{dt}(S + I + R) + \frac{(\alpha + 2\mu)}{\beta} \frac{(I - I^*)}{I} \frac{dI}{dt} \]

\[
+ \frac{(\alpha + 2\mu)}{\kappa} (R - R^*) \frac{dR}{dt},
\]

\[
= [(S - S^*) + (I - I^*) + (R - R^*)](\Lambda - \mu(S + I + R) - \alpha I)
\]

\[
+ \frac{(\alpha + 2\mu)}{\beta} \frac{(I - I^*)}{I} (\beta SI - (\kappa + \mu + \alpha)I)
\]

\[
+ \frac{(\alpha + 2\mu)}{\kappa} (R - R^*)(\kappa I - (\gamma + \mu)R).
\]

Using

\[
\Lambda = \mu(S^* + I^* + R^*) + \alpha I^*,
\]

\[
\beta S^* = (\kappa + \mu + \alpha),
\]

\[
0 = (\mu + \gamma)R^* - \kappa I^*,
\]

to rewrite this, we get

\[
L' = -[(S - S^*) + (I - I^*) + (R - R^*)] \mu [(S - S^*) + (R - R^*)] + (\alpha + \mu)(I - I^*)
\]

\[
+ (\alpha + 2\mu)(S - S^*)(I - I^*)
\]

\[
+ \frac{(\alpha + 2\mu)}{\kappa} (R - R^*)(\kappa(I - I^*) - (\mu + \gamma)(R - R^*)),
\]

\[
= -\mu[(S - S^*) + (R - R^*)]^2 - (\alpha + \mu)(I - I^*)^2
\]

\[
- \frac{(\gamma + \mu)(\alpha + 2\mu)}{\kappa} (R - R^*)^2.
\]

Hence \( L' \) is negative. Note that, \( L' = 0 \) if and only if \( S = S^* \), \( I = I^* \) and \( R = R^* \). Therefore the largest compact invariant set in \( \{(S, I, R) \in \Gamma : L' = 0\} \) is the singleton \( \{E^*\} \), where \( E^* \) is the endemic equilibrium. LaSalle’s invariant principle [5] then implies that \( E^* \) is globally asymptotically stable in the interior of \( \Gamma \).

4 **Other Lyapunov functions**

In this section, we present some examples of the non-uniqueness of Lyapunov functions in classic SIS, SIR and SIRS epidemic models. The Lyapunov functions are constructed to establish the global stability of the endemic equilibriums.
For SIS epidemic model Korobeinikov and Wake in [4] reporting the logarithmic Lyapunov function:

\[ \hat{V}(S, I) = \left( S - S^* - S^* \ln \frac{S}{S^*} \right) + \frac{(\alpha + \mu)}{(\alpha + \gamma + \mu)} \left( I - I^* - I^* \ln \frac{I}{I^*} \right), \]

and the time derivative of function \( \hat{V} \) is given by

\[ \hat{V}'(S, I) = -(\Lambda + \gamma I) \frac{(S - S^*)^2}{SS^*}. \]

We construct the Lyapunov function of the combination of common quadratic and logarithmic functions for epidemic model (1):

\[ \tilde{V}(S, I) = \frac{1}{2} (S - S^*)^2 + \frac{(\alpha + \mu)}{\beta} \left( I - I^* - I^* \ln \frac{I}{I^*} \right), \]

and the derivative of \( \tilde{V} \) with respect to \( t \) along solution curves is given by

\[ \tilde{V}'(S, I) = -(\mu + \beta I)(S - S^*)^2. \]

We construct another Lyapunov function of the combination of composite quadratic and logarithmic functions for systems (1):

\[
\begin{align*}
\mathcal{V}(S, I) &= \frac{1}{2} [(S - S^*) + (I - I^*)]^2 + \frac{(\alpha + 2\mu)S^*}{2\gamma} \left( S - S^* - S^* \ln \frac{S}{S^*} \right) \\
&\quad + \frac{(\alpha + 2\mu)S^*}{\gamma} \left( I - I^* - I^* \ln \frac{I}{I^*} \right),
\end{align*}
\]

and the time derivative of \( \mathcal{V} \) along the solutions of system (1), is

\[
\mathcal{V}'(S, I) = -\mu(S - S^*)^2 - (\alpha + 2\mu) \left( \frac{\Lambda}{\gamma} + I \right) \frac{(S - S^*)^2}{S} - (\alpha + \mu)(I - I^*)^2.
\]

In [1] and [4] proved the global stability of endemic equilibrium of SIR epidemic model in SI plane may be constructed of logarithmic functions

\[ \hat{L}(S, I) = \left( S - S^* - S^* \ln \frac{S}{S^*} \right) + \left( I - I^* - I^* \ln \frac{I}{I^*} \right), \]
and the time derivative of function $\hat{L}$ is given by

$$\hat{L}'(S, I) = -\Lambda \frac{(S - S^*)^2}{SS^*}.$$ 

For $SIR$ epidemic model Mena-Lorca and Hethcote in [6] reported the common quadratic and logarithmic functions in $SI$ plane:

$$\bar{L}(S, I) = \frac{1}{2}(S - S^*)^2 + S^* \left( I - I^* - I^* \ln \frac{I}{I^*} \right),$$

and the Lyapunov derivative is,

$$\bar{L}'(S, I) = -(\mu + \beta I)(S - S^*)^2.$$ 

We construct another Lyapunov function of the combination of composite quadratic and logarithmic functions for $SIR$ epidemic model in $SI$ plane:

$$\hat{L}(S, I) = \frac{1}{2} \left[ (S - S^*) + (I - I^*) \right]^2 + \frac{(\alpha + \kappa + 2\mu)}{\beta} \left( I - I^* - I^* \ln \frac{I}{I^*} \right),$$

and the derivative of $\hat{L}$ along the solutions curve, is given by the expression

$$\hat{L}'(S, I) = -\mu(S - S^*)^2 - (\alpha + \kappa + \mu)(I - I^*)^2.$$ 

For $SIRS$ model we construct other Lyapunov function of the combination of composite quadratic, common quadratic and logarithmic functions for proved the global stability of endemic equilibrium:

$$J(S, I, R) = \frac{1}{2} \left[ (S - S^*) + (I - I^*) + (R - R^*) \right]^2 + \frac{\mu}{\gamma} (S - S^*)^2$$

$$+ \left( \frac{(\alpha + 2\mu)}{\beta} + \frac{2\mu S^*}{\gamma} \right) \left( I - I^* - I^* \ln \frac{I}{I^*} \right) + \frac{(\alpha + 2\mu)}{2\kappa} (R - R^*)^2,$$

and the time derivative of $\overline{V}$ along the solutions of system (3), is

$$J'(S, I, R) = -\frac{\mu}{\gamma} \left( \gamma + 2\mu + 2\beta I \right) (S - S^*)^2 - (\alpha + \mu)(I - I^*)^2$$

$$- \left( \mu + \frac{(\mu + \gamma)(\alpha + 2\mu)}{\kappa} \right) (R - R^*)^2.$$
5 Conclusions

Our main interest is to investigate the construction of Lyapunov function to prove the global stability of the equilibria in epidemiological models with varying population size.

The results of this work indicate that the suitable combination of composite quadratic, common quadratic and logarithmic functions

\[ W(x_1, x_2, \ldots, x_n) = \frac{c_2}{2} \left[ \sum_{i=1}^{n} (x_i - x_i^*) \right]^2, \]

\[ V(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} c_i (x_i - x_i^*)^2 \]

and

\[ L(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} c_i (x_i - x_i^* - x_i^* \ln \frac{x_i}{x_i^*}), \]

respectively, can be especially useful for others epidemic models.

References


